

Midterm Solutions

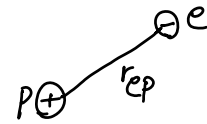
Note Title

3/18/2008

①

$$\hat{H} = \hat{T} + \hat{U}$$

$$\hat{H} = \underbrace{-\frac{\hbar^2}{2m} \nabla_e^2}_{T_e} - \underbrace{\frac{\hbar^2}{2m} \nabla_p^2}_{T_p} - \underbrace{\frac{e^2}{4\pi\epsilon_0 r_{ep}}}_{U_{ep}}$$



②

$$L = 3 \frac{\lambda}{2} \quad \text{2nd excited} \Rightarrow \lambda = \frac{2}{3} L = 100 \text{ nm}$$

$$L = \lambda \quad \text{1st excited}$$

$$L = \frac{\lambda}{2} \quad \text{Ground}$$

A diagram of a string of length L fixed at both ends. Three standing wave patterns are shown: the ground state (one antinode), the first excited state (two antinodes), and the second excited state (three antinodes). The length L is indicated at the bottom.

$$\textcircled{3} \quad E = \frac{\hbar^2 \pi^2}{L^2} (n_x^2 + n_y^2 + n_z^2)$$

First state (Ground): $E_{100} = E_{101} = E_{001} = \frac{\hbar^2 \pi^2}{L^2} (1)$

First excited state: $E_{110} = E_{101} = E_{011} = \frac{\hbar^2 \pi^2}{L^2} (2) \Rightarrow 3 \text{ fold degen.}$

④

$$E_b = (4t)(3) = 12t$$

$\uparrow \quad \uparrow$
 for 1D 3D

⑤

$$m^* = \frac{\hbar^2}{\frac{\partial^2 E}{\partial k^2}}$$

$$E = \frac{E_b}{2} (1 - \cos kL)$$

$$\frac{\partial E}{\partial k} = \frac{E_b}{2} L \sin kL$$

$$\frac{\partial^2 E}{\partial k^2} = \frac{E_b}{2} L^2 \cos kL$$

$$\Rightarrow m^* = \frac{2\hbar^2}{E_b L^2 \cos kL}$$

at the band edge: $E \approx 0 \Rightarrow k \approx 0 \Rightarrow$

$$m^* \approx \frac{2\hbar^2}{E_b L^2} = \frac{(2)(6.58 \times 10^{-16})^2}{(1.0848 \times 10^{-18})(0.565 \times 10^{-9})^2}$$
$$= 6.42 \times 10^{-32} \text{ kg}$$

$$\text{or: } \frac{m^*}{m_e} = \frac{6.42 \times 10^{-32}}{9.1 \times 10^{-31}} \approx 0.07$$

⑥

$$|\psi\rangle = \sum_n b_n |n\rangle$$

$$\langle m | \psi \rangle = \sum_n b_n \langle m | n \rangle = \sum_n b_n \delta_{mn}$$
$$= b_m$$

$$\Rightarrow b_n = \langle n | \psi \rangle$$

⑦

$$[\hat{x} + \hat{y}, \hat{p}_x + \hat{p}_y] = (\hat{x} + \hat{y})(\hat{p}_x + \hat{p}_y) - (\hat{p}_x + \hat{p}_y)(\hat{x} + \hat{y})$$

$$= \underbrace{[\hat{x}, \hat{p}_x]}_{i\hbar} + \underbrace{[\hat{x}, \hat{p}_y]}_0 + \underbrace{[\hat{y}, \hat{p}_x]}_0 + \underbrace{[\hat{y}, \hat{p}_y]}_{i\hbar}$$

$$= 2i\hbar$$

⑧

$$D(E) dE = 2 \times \frac{dk}{2\pi}$$

$$D(E) = \frac{1}{\pi} \frac{dk}{dE} \quad E = \frac{\hbar^2 k^2}{2m} \Rightarrow \frac{dE}{dk} = \frac{\hbar^2 k}{m} \Rightarrow$$

$$D(E) = \frac{1}{\pi} \frac{m}{\hbar^2} \frac{1}{k} = \frac{m}{\pi \hbar^2} \frac{\hbar}{\sqrt{2mE}}$$

$$D(E) = \frac{m^{1/2}}{2^{1/2} \pi \hbar} \frac{1}{\sqrt{E}}$$

$$\begin{aligned}
 \textcircled{9} \quad \langle n | \hat{x} | n+1 \rangle &= \left(\frac{\hbar}{2m\omega} \right)^{1/2} \langle n | a + a^\dagger | n+1 \rangle \\
 &= \left(\frac{\hbar}{2m\omega} \right)^{1/2} \left[\langle n | a | n+1 \rangle + \langle n | a^\dagger | n+1 \rangle \right] \\
 &= \left(\frac{\hbar}{2m\omega} \right)^{1/2} \left[\sqrt{n+1} \underbrace{\langle n | n \rangle}_1 + \sqrt{n+2} \underbrace{\langle n | n+2 \rangle}_0 \right] \\
 &= \left(\frac{\hbar}{2m\omega} \right)^{1/2} \sqrt{n+1}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{10} \quad \hat{a}^\dagger |\psi_0\rangle &= |\psi_1\rangle & p_x &= -i\hbar \frac{\partial}{\partial x} \\
 \Rightarrow \psi_1(x) &= \hat{a}^\dagger \psi_0(x) = \left(\frac{m\omega}{2\hbar} \right)^{1/2} \left(x - \frac{i p_x}{m\omega} \right) \psi_0(x) \\
 &= \left(\frac{m\omega}{2\hbar} \right)^{1/2} \left(\frac{m\omega}{\hbar} \right)^{1/4} \left(x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) e^{-\frac{m\omega}{2\hbar} x^2} \\
 &= \left(\frac{m\omega}{\hbar} \right)^{3/4} \frac{1}{2^{1/2} \pi^{1/4}} \left[x - \left(\frac{\hbar}{m\omega} \right) \left(-\frac{m\omega}{\hbar} \right) x \right] e^{-\frac{m\omega}{2\hbar} x^2} \\
 &= \frac{1}{2^{1/2} \pi^{1/4}} \left(\frac{m\omega}{\hbar} \right)^{3/4} (2x) e^{-\frac{m\omega}{2\hbar} x^2} \\
 \psi_1 &= \frac{2^{1/2}}{\pi^{1/4}} \left(\frac{m\omega}{\hbar} \right)^{3/4} x e^{-\frac{m\omega}{2\hbar} x^2}
 \end{aligned}$$

