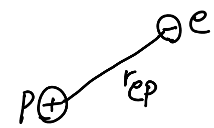
Midterm Solutions

① $\hat{H} = \hat{T} + \hat{U}$



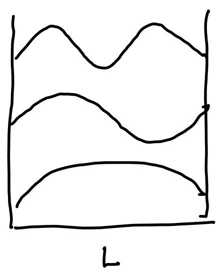
$$\hat{H} = \underbrace{-\frac{\hbar^2}{2m}\nabla_e^2}_{T_e} \underbrace{-\frac{\hbar^2}{2m}\nabla_p^2}_{T_p} \underbrace{-\frac{e^2}{4\pi\varepsilon_0 r_{ep}}}_{U_{ep}}$$

②

$L = 3\frac{\lambda}{2}$ — 2^{nd} excited $\Rightarrow \lambda = \frac{2}{3}L = 100\,nm$

$L = \lambda$ — 1^{st} excited

$L = \frac{\lambda}{2}$ — Ground



③ $E = \dfrac{\hbar^2\pi^2}{L^2}\left(n_x^2 + n_y^2 + n_z^2\right)$

First state (Ground) : $E_{100} = E_{101} = E_{001} = \dfrac{\hbar^2\pi^2}{L^2}(1)$

First excited state : $E_{110} = E_{101} = E_{011} = \dfrac{\hbar^2\pi^2}{L^2}(2)$ $\Rightarrow$ 3 fold degen.

④ $E_b = \underbrace{(4t)}_{\text{for 1D}}\underbrace{(3)}_{\text{3D}} = 12t$

⑤ $m^* = \dfrac{\hbar^2}{\partial^2 E/\partial k^2}$

$$E = \frac{E_b}{2}(1 - \cos kL)$$

$$\frac{\partial E}{\partial k} = \frac{E_b}{2} L \sin kL$$

$$\frac{\partial^2 E}{\partial k^2} = \frac{E_b}{2} L^2 \cos kL$$

$$\Rightarrow m^* = \frac{2\hbar^2}{E_b L^2 \cos kL}$$

at the band edge: $E \simeq 0 \Rightarrow k \simeq 0 \Rightarrow$

$$m^* \simeq \frac{2\hbar^2}{E_b L^2} = \frac{(2)(6.58\times10^{-16})^2}{(1.0848\times10^{-18})(0.565\times10^{-9})^2}$$

$$= 6.42\times10^{-32}\ \text{kg}$$

or: $\frac{m^*}{m_e} = \frac{6.42\times10^{-32}}{9.1\times10^{-31}} \simeq 0.07$

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$$|\psi\rangle = \sum_n b_n |n\rangle$$

$$\langle m|\psi\rangle = \sum_n b_n \langle m|n\rangle = \sum_n b_n \delta_{mn}$$

$$= b_m$$

$$\Rightarrow b_n = \langle n|\psi\rangle$$

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$$[\hat{x}+\hat{y},\ \hat{P}_x+\hat{P}_y] = (\hat{x}+\hat{y})(\hat{P}_x+\hat{P}_y) - (\hat{P}_x+\hat{P}_y)(\hat{x}+y)$$

$$= \underbrace{[\hat{x},\hat{P}_x]}_{i\hbar} + \underbrace{[\hat{x},\hat{P}_y]}_{0} + \underbrace{[\hat{y},\hat{P}_x]}_{0} + \underbrace{[\hat{y},\hat{P}_y]}_{i\hbar}$$

$$= 2i\hbar$$

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$$D(E)\,dE = 2\times\frac{dk}{2\pi}$$

$$D(E) = \frac{1}{\pi}\frac{dk}{dE} \qquad E = \frac{\hbar^2k^2}{2m} \Rightarrow \frac{dE}{dk} = \frac{\hbar^2 k}{m} \Rightarrow$$

$$D(E) = \frac{1}{\pi}\frac{m}{\hbar^2}\frac{1}{k} = \frac{m}{\pi\hbar^2}\frac{\hbar}{\sqrt{2mE}}$$

$$D(E) = \frac{m^{1/2}}{2^{1/2}\pi\hbar}\frac{1}{\sqrt{E}}$$

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$$\langle n|\hat{x}|n+1\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \langle n|a+a^{+}|n+1\rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2} \left[\langle n|a|n+1\rangle + \langle n|a^{+}|n+1\rangle\right]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2} \left[\sqrt{n+1}\,\underbrace{\langle n|n\rangle}_{1} + \sqrt{n+2}\,\underbrace{\langle n|n+2\rangle}_{0}\right]$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2} \sqrt{n+1}$$

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$$\hat{a}^{+}|\psi_0\rangle = |\psi_1\rangle$$

$$P_x = -i\hbar\frac{\partial}{\partial x}$$

$$\Rightarrow \quad \psi_1(x) = \hat{a}^{+}\psi_0(x) = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x - \frac{iP_x}{m\omega}\right)\psi_0(x)$$

$$= \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \left(\frac{m\omega}{\hbar}\right)^{3/4}\frac{1}{2^{1/2}\pi^{1/4}}\left[x - \left(\frac{\hbar}{m\omega}\right)\left(\frac{-m\omega}{\hbar}\right)x\right] e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{2^{1/2}\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{3/4}(2x)\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_1 = \frac{2^{1/2}}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{3/4} x\, e^{-\frac{m\omega}{2\hbar}x^2}$$